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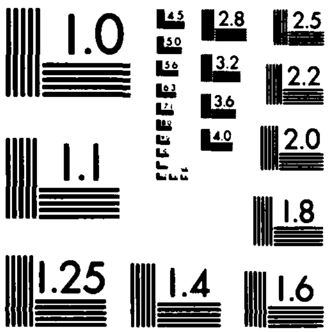

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TECHNICAL REPORT RG-83-6

EXPRESSIONS FOR THE EXACT STATE TRANSITION MATRIX OF A NINE-STATE TARGET MODEL

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U.S. ARMY MISSILE COMMAND
Redstone Arsenal, Alabama 35809

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Several expressions have been developed for the exact state transition matrix associated with a time-invariant, 9x9 target state matrix. Each form results from a different combination of allowable parameter values representing the specified bandwidths in the differential equations describing the target acceleration components along three orthogonal coordinate axes.

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CONTENTS

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I. INTRODUCTION

The Analytic Sciences Corporation (TASC) recently performed a study (see Reference 1) for MICOM on the implementation of a strapdown inertial updated midcourse guidance scheme for application to Short Range Air Defense type missiles. Part of this effort involved design of an Extended Kalman Filter (EKF) to process the radar measurement data received from a "quiet radar," i.e., track on scan radar, as it scans a target. The target system dynamic model was assumed to be linear in a rectangular coordinate system and the target state vector was defined as

$$x = (x, y, z, \dot{x}, \dot{y}, \dot{z}, \ddot{x}, \ddot{y}, \ddot{z}), \tag{1}$$

where (x, y, z) denote position, $(\dot{x}, \dot{y}, \dot{z})$ denote velocity, and $(\ddot{x}, \ddot{y}, \ddot{z})$ denote acceleration. Since the target accelerations cannot be directly measured, they are modeled as first-order Markov processes given by the differential equations

$$\begin{aligned} \dddot{x} &= -\lambda_x \ddot{x} + w_x \\ \dddot{y} &= -\lambda_y \ddot{y} + w_y \\ \dddot{z} &= -\lambda_z \ddot{z} + w_z \end{aligned} \tag{2}$$

where λ_x, λ_y, λ_z are specified bandwidths (nominally equal and ≥ 0) and w_x, w_y, w_z are white Gaussian noise processes with zero means and specified spectral densities.

The dynamic system model for the target tracking filter which processes the radar measurement data was given (see Reference 1) as

$$\dot{x} = Fx + w \tag{3}$$

where F is a time-invariant 9x9 matrix and w is the system process noise vector. Expressing equation (3) in expanded form results in

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \hline \ddot{x} \\ \ddot{y} \\ \ddot{z} \\ \hline \dddot{x} \\ \dddot{y} \\ \dddot{z} \end{pmatrix} = \left[\begin{array}{c|ccc|ccc} & 1 & 0 & 0 & & & \\ 0 & 0 & 1 & 0 & & 0 & \\ & 0 & 0 & 1 & & & \\ \hline & & & & 1 & 0 & 0 \\ 0 & & 0 & & 0 & 1 & 0 \\ & & & & 0 & 0 & 1 \\ \hline & & & & -\lambda_x & 0 & 0 \\ 0 & & 0 & & 0 & -\lambda_y & 0 \\ & & & & 0 & 0 & -\lambda_z \end{array}\right] \begin{pmatrix} x \\ y \\ z \\ \hline \dot{x} \\ \dot{y} \\ \dot{z} \\ \hline \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ \hline 0 \\ 0 \\ 0 \\ \hline w_x \\ w_y \\ w_z \end{pmatrix} . \tag{4}$$

In order to obtain a solution for x from Equation (3) an expression must be found for the system state transition matrix defined by

$$\Phi = e^{F\Delta t}, \tag{5}$$

where Δt is the time interval (assumed to be constant) between measurements and F is defined as in Equation (4).

There are several methods (see Reference 2) which may be used to evaluate a matrix exponential of the form e^{Ft}. The two more commonly used are:
(a) expansion of e^{Ft} into a power series in t as

$$e^{Ft} = I + Ft + \frac{(Ft)^2}{2} + \frac{(Ft)^3}{3!} + \text{——} + \frac{(Ft)^n}{n!} + \text{——} , \tag{6}$$

and (b) the Laplace transform method whereby e^{Ft} is found as

$$e^{Ft} = \mathcal{L}^{-1}\{(sI - F)^{-1}\}. \tag{7}$$

As the order of the F matrix increases, the computations involved in obtaining an exact evaluation of e^{Ft} using Equation (7) become extremely burdensome. As a consequence, TASC chose to approximate $e^{F\Delta t}$ by using the first two terms of a power series expansion in Δt, i.e.,

$$e^{F\Delta t} \approx I + F\Delta t, \tag{8}$$

which is very simple to calculate.

An exact expression will be developed, in this report, for the 9-state target model state transition matrix defined by Equation (5) with the matrix F of Equation (4). In the process, another method (based on the LaGrange-Sylvester interpolation polynomial (see References 2 and 3) which can be used to evaluate e^{Ft}, and which enables one to arrive at an exact expression for e^{Ft} without requiring matrix inversion, will be illustrated.

II. MATHEMATICAL BACKGROUND

The method to be used in evaluating e^{Ft} will be presented in this section, and an example will be worked to illustrate application of this method in comparison with the Laplace transform method. First though, some terms which must be defined are as follows:

(a) Annihilating Polynomial: An annihilating polynomial of a square matrix F is a scalar polynomial $f(\lambda)$ for which $f(F) = 0$.

(b) Monic Polynomial: A monic polynomial is a scalar polynomial $f(\lambda) = c_o\lambda^k + c_1\lambda^{k-1} + \ldots. + c_k$ in which $c_o = 1$.

(c) Minimal Polynomial: A minimal polynomial of a matrix F is the monic annihilating polynomial of least degree.

(d) Spectrum: The spectrum of a matrix F is the set of characteristic values of F, i.e., the set of all scalars λ for which $F-\lambda I$ is not invertible.

A function $f(\lambda)$ is said to be defined on the spectrum of a matrix F if the values of the function $f(\lambda)$ on the spectrum of F exist, i.e., if the m numbers

$$f(\lambda_k),\ f'(\lambda_k),\ \text{---}\ ,\ f^{(m_k-1)}(\lambda_k) \qquad (k = 1,\ 2,\ \text{---}\ ,\ s)$$

have meaning, where s is the number of distinct characteristic values of F and m_k is the multiplicity of the k-th characteristic value in the minimal polynomial (see Reference 3).

The Lagrange-Sylvester interpolation polynomial for a function $f(\lambda)$ defined on the spectrum of a matrix F is a polynomial $r(\lambda)$ defined as

$$r(\lambda) = \sum_{k=1}^{s} \left(\sum_{j=1}^{m_k} a_{kj}(\lambda - \lambda_k)^{j-1} \right) \Psi_k(\lambda), \tag{9}$$

where

$$a_{kj} = \frac{1}{(j-1)!} \left[\frac{d^{j-1}}{d\lambda^{j-1}} \left(\frac{f(\lambda)}{\Psi_k(\lambda)} \right) \right]_{\lambda = \lambda_k} \quad (j = 1,\ 2,\ \text{---},\ m_k;\ k = 1,\ 2,\ \text{---},\ s), \tag{10}$$

$\Psi_k(\lambda)$ is the minimal polynomial of F with the λ_k term removed, and $r(\lambda)$ satisfies the conditions $r(\lambda_k) = f(\lambda_k)$, $r'(\lambda_k) = f'(\lambda_k)$, ---, $r^{(m_k-1)}(\lambda_k) = f^{(m_k-1)}(\lambda_k)$, $(k = 1,\ 2,\ \text{---},\ s)$. The polynomial $r(\lambda)$ assumes the same values on the spectrum of F as does $f(\lambda)$ and one has $f(F) = r(F)$.

If the equation for the coefficients (Equation (10)) is expanded and substituted into Equation (9), Equation (9) can be expressed (see Reference 3) in the form

$$r(\lambda) = \sum_{k=1}^{s} \sum_{j=1}^{m_k} f^{(j-1)}(\lambda_k)\phi_{kj}(\lambda) \tag{11}$$

where the $\phi_{kj}(\lambda)$ represent polynomials in λ, of degree less than m, which are completely determined when $\Psi(\lambda)$ is given and are independent of the choice of $f(\lambda)$. Equation (11) can be rewritten to obtain the fundamental formula for $f(F)$ (see Reference 3):

$$f(F) = \sum_{k=1}^{s} \sum_{j=1}^{m_k} f^{(j-1)}(\lambda_k) Z_{kj} \tag{12}$$

where $Z_{kj} = \phi_{kj}(F)$ $(j = 1,\ 2,\ \text{---},\ m_k;\ k = 1,\ 2,\ \text{---},\ s)$. As for the $\phi_{kj}(\lambda)$ terms of Equation (11), the Z_{kj} matrices are completely determined by F and are independent of the choice of $f(\lambda)$. The Z_{kj} matrices are called (see Reference 3) the components of the given matrix F.

There are several methods by which the Z_{kj} matrices can be evaluated. The one which will be illustrated here is the method whereby certain simple polynomials for $f(\lambda)$ are successively substituted into the fundamental formula given by Equation (12). From the linear equations so generated the Z_{kj} matrices can be determined. For comparison, the Laplace transformation method for finding e^{Ft} will also be illustrated.

Example 1: <u>Lagrange-Sylvester Method</u>

Suppose one wishes to find an expression for e^{Ft} where F is given as

$$F = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{bmatrix}. \tag{13}$$

For this given matrix F one has $\Delta(\lambda) = \Psi(\lambda) = (\lambda-1)^2\ (\lambda-2)$, thus, $\lambda_1 = 1$, $\lambda_2 = 2$, $m_1 = 2$ and $m_2 = 1$. Checking to see that $f(\lambda) = e^{\lambda t}$ is defined on the spectrum of F it is seen that, for the root $\lambda_1 = 1$ with $m_1 = 2$,

$$f(\lambda_1) = e^t \text{ exists and}$$

$$f'(\lambda_1) = e^t \text{ exists.}$$

Also, for $\lambda_2 = 2$ with $m_2 = 1$ it is seen that $f(\lambda_2) = e^{2t}$ exists. Hence, $f(\lambda) = e^{\lambda t}$ is defined on the spectrum of F.

Next, making use of Equation (12), f(F) can be written as

$$f(F) = e^{Ft} = f(1)Z_{11} + f'(1)Z_{12} + f(2)Z_{21} \tag{14}$$

and the components of F can be determined as follows. First, take $r(\lambda)$ to be $r(\lambda) = (\lambda-1)^2$. For this choice of $r(\lambda)$, one has

$$r(F) = (F-I)^2 = Z_{21}. \tag{15}$$

If one next takes $r(\lambda)$ as $r(\lambda) = (\lambda-1)(\lambda-2)$ then one has

$$r(F) = (F-I)(F-2I) = (-1)\ Z_{12}$$

and, therefore,

$$Z_{12} = -\ (F-I)(F-2I)\ . \tag{16}$$

Finally, take $r(\lambda)$ as $r(\lambda) = 1$. In this case, one has

$$r(F) = I = Z_{11} + Z_{21}$$

and Z_{11} is found to be

$$Z_{11} = I - Z_{21} = I - (F-I)^2. \tag{17}$$

If Equation (15), (16) and (17) are substituted into Equation (14), the resultant expression for e^{Ft} is found as

$$\begin{aligned} e^{Ft} &= e^{\lambda_1 t}Z_{11} + te^{\lambda_1 t}Z_{12} + e^{\lambda_2 t}Z_{21} \\ &= [I-(F-I)^2]e^t - [(F-I)(F-2I)]te^t + (F-I)^2\ e^{2t}\ . \end{aligned} \tag{18}$$

Performing the matrix multiplication indicated in Equation (18) will result in the following expression for e^{Ft}:

$$e^{Ft} = \begin{bmatrix} 0 & 2 & -1 \\ -2 & 5 & -2 \\ -4 & 8 & -3 \end{bmatrix} e^t - \begin{bmatrix} 2 & -3 & 1 \\ 2 & -3 & 1 \\ 2 & -3 & 1 \end{bmatrix} te^t + \begin{bmatrix} 1 & -2 & 1 \\ 2 & -4 & 2 \\ 4 & -8 & 4 \end{bmatrix} e^{2t}$$

$$= \begin{bmatrix} -2te^t + e^{2t} & (2+3t)e^t - 2e^{2t} & (-1-t)e^t + e^{2t} \\ (-2-2t)e^t + 2e^{2t} & (5+3t)e^t - 4e^{2t} & (-2-t)e^t + 2e^{2t} \\ (-4-2t)e^t + 4e^{2t} & (8+3t)e^t - 8e^{2t} & (-3-t)e^t + 4e^{2t} \end{bmatrix}. \quad (19)$$

Example 2: <u>Laplace Transform Method</u>

To evaluate e^{Ft} using the Laplace transform method given by Equation (7), the sequence of operations to be performed would be as follows. First, form the matrix (sI-F) as

$$(sI-F) = \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ -2 & 5 & s-4 \end{bmatrix}.$$

The determinant of (sI-F) is $\Delta(s) = (s-1)^2\,(s-2)$. The matrix inverse, $(sI-F)^{-1}$, must next be calculated and is found to be

$$(sI-F)^{-1} = \frac{1}{(s-1)^2(s-2)} \begin{bmatrix} s^2 - 4s + 5 & s-4 & 1 \\ 2 & s(s-4) & s \\ 2s & -5s + 2 & s^2 \end{bmatrix}. \quad (20)$$

The expression for e^{Ft} is calculated by taking the inverse Laplace transform of each term of $(sI-F)^{-1}$. If Equation (20) is expanded using partial fractions, the result is given by

$$(sI-F)^{-1} = \begin{bmatrix} -\frac{2}{(s-1)^2} + \frac{1}{s-2} & \frac{3}{(s-1)^2} + \frac{2}{s-1} - \frac{2}{s-2} & \left(-\frac{1}{(s-1)^2} - \frac{1}{s-1} + \frac{1}{s-2}\right) \\ -\frac{2}{(s-1)^2} - \frac{2}{s-1} + \frac{2}{s-2} & \frac{3}{(s-1)^2} + \frac{5}{s-1} - \frac{4}{s-2} & \left(-\frac{1}{(s-1)^2} - \frac{2}{s-1} + \frac{2}{s-2}\right) \\ -\frac{2}{(s-1)^2} - \frac{4}{s-1} + \frac{4}{s-2} & \frac{3}{(s-1)^2} + \frac{8}{s-1} - \frac{8}{s-2} & \left(-\frac{1}{(s-1)^2} - \frac{3}{s-1} + \frac{4}{s-2}\right) \end{bmatrix} \quad (21)$$

and taking the inverse Laplace transform of each term then gives the final expression for e^{Ft} as

$$e^{Ft} = \begin{bmatrix} -2te^{t} + e^{2t} & (3t+2)e^{t} - 2e^{2t} & (-t-1)e^{t} + e^{2t} \\ (-2t-2)e^{t} + 2e^{2t} & (3t+5)e^{t} - 4e^{2t} & (-t-2)e^{t} + 2e^{2t} \\ (-2t-4)e^{t} + 4e^{2t} & (3t+8)e^{t} - 8e^{2t} & (-t-3)e^{t} + 4e^{2t} \end{bmatrix}. \qquad (22)$$

Comparing Equations (22) and (19), it is seen that both approaches result in the same answer; however, evaluation of e^{Ft} using the Laplace transform method involves more of a computation burden than does the method using matrix components, even for a 3x3 matrix.

III. TARGET MODEL STATE TRANSITION MATRIX EVALUATION

A. Introduction

In this section the matrix exponential e^{Ft}, with F as defined in Equation (4), will be evaluated to obtain an exact expression for the target state transition matrix for several different combinations of λ_x, λ_y, λ_z. The first step in the process is to evaluate the determinant of $(F-\lambda I)$ as

$$|F-\lambda I| = \left| \begin{array}{c|c|ccc} -\lambda I & I & & 0 & \\ \hline 0 & -\lambda I & & I & \\ \hline & & -\lambda_x-\lambda & 0 & 0 \\ 0 & 0 & 0 & -\lambda_y-\lambda & 0 \\ & & 0 & 0 & -\lambda_z-\lambda \end{array} \right|$$

$$= \lambda^6[\lambda^3 + (\lambda_x+\lambda_y+\lambda_z)\lambda^2 + (\lambda_x\lambda_y+\lambda_x\lambda_z+\lambda_y\lambda_z)\lambda + \lambda_x\lambda_y\lambda_z]. \qquad (23)$$

The characteristic values, therefore, are given by

$$\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = \lambda_6 = 0,\ \lambda_7 = -\lambda_x,\ \lambda_8 = -\lambda_y,\ \lambda_9 = -\lambda_z. \qquad (24)$$

The next step is to determine the minimal polynomial of F. The minimal polynomial of a matrix will contain each of the distinct characteristic values of the matrix with a multiplicity not greater than each root has in the characteristic polynomial. The candidate choices for the minimal polynomial of F are thus:

$$\begin{array}{l} (\lambda+\lambda_x)(\lambda+\lambda_y)(\lambda+\lambda_z)\lambda \\ (\lambda+\lambda_x)(\lambda+\lambda_y)(\lambda+\lambda_z)\lambda^2 \\ \qquad\vdots \\ (\lambda+\lambda_x)(\lambda+\lambda_y)(\lambda+\lambda_z)\lambda^6 \end{array} \qquad (25)$$

and the actual minimal polynomial will depend on the values chosen for λ_x, λ_y, λ_z.

B. Case With λ_x, λ_y, λ_z Positive, Unequal

Using each of the expressions in Equation (25) in turn, $\Psi(F)$ is evaluated to determine which expression will result in $\Psi(F) \equiv 0$. Upon performing this procedure, the minimal polynomial (for the λ's of this section) is found to be

$$\Psi(\lambda) = \lambda^2(\lambda+\lambda_x)(\lambda+\lambda_y)(\lambda+\lambda_z). \tag{26}$$

The fundamental formula for f(F), as given by Equation (12), will therefore be

$$\begin{aligned} f(F) &= f(\lambda_1)Z_{11} + f'(\lambda_1)Z_{12} + f(\lambda_2)Z_{21} + f(\lambda_3)Z_{31} + f(\lambda_4)Z_{41} \\ &= f(0)Z_{11} + f'(0)Z_{12} + f(-\lambda_x)Z_{21} + f(-\lambda_y)Z_{31} + f(-\lambda_z)Z_{41}. \end{aligned} \tag{27}$$

For $f(\lambda) = e^{\lambda t}$ one has $f(F) = e^{Ft}$ so that, using Equation (27),

$$\begin{aligned} e^{Ft} &= e^{\lambda_1 t}Z_{11} + te^{\lambda_1 t}Z_{12} + e^{\lambda_2 t}Z_{21} + e^{\lambda_3 t}Z_{31} + e^{\lambda_4 t}Z_{41} \\ &= (Z_{11}+tZ_{12}) + e^{-\lambda_x t}Z_{21} + e^{-\lambda_y t}Z_{31} + e^{-\lambda_z t}Z_{41}. \end{aligned} \tag{28}$$

The components of F can now be found by choosing simple polynomials for $r(\lambda)$. This process is illustrated in the following steps for $r(\lambda) = r(\lambda_1)Z_{11} + r'(\lambda_1)Z_{12} + r(\lambda_2)Z_{21} + r(\lambda_3)Z_{31} + r(\lambda_4)Z_{41}$.

1. $r(\lambda) = \lambda(\lambda+\lambda_x)(\lambda+\lambda_y)(\lambda+\lambda_z)$

   $r'(\lambda) = 4\lambda^3 + 3(\lambda_x+\lambda_y+\lambda_z)\lambda^2 + 2(\lambda_x\lambda_y+\lambda_x\lambda_z+\lambda_y\lambda_z)\lambda + \lambda_x\lambda_y\lambda_z$

   $F(F+\lambda_x I)(F+\lambda_y I)(F+\lambda_z I) = (0)Z_{11} + \lambda_x\lambda_y\lambda_z Z_{12} + (0)Z_{21} + (0)Z_{31} + (0)Z_{41}$

   $$Z_{12} = \frac{1}{\lambda_x\lambda_y\lambda_z}\left[F(F+\lambda_x I)(F+\lambda_y I)(F+\lambda_z I)\right] \tag{29}$$

2. $r(\lambda) = (\lambda+\lambda_x)(\lambda+\lambda_y)(\lambda+\lambda_z)$

   $r'(\lambda) = 3\lambda^2 + 2(\lambda_x+\lambda_y+\lambda_z)\lambda + (\lambda_x\lambda_y+\lambda_x\lambda_z+\lambda_y\lambda_z)$

   $(F+\lambda_x I)(F+\lambda_y I)(F+\lambda_z I) = \lambda_x\lambda_y\lambda_z Z_{11} + (\lambda_x\lambda_y+\lambda_x\lambda_z+\lambda_y\lambda_z)Z_{12} + (0)Z_{21} + (0)Z_{31} + (0)Z_{41}$

   $$Z_{11} = (\lambda_x\lambda_y\lambda_z)^{-1}\left[I - \left(\frac{\lambda_x\lambda_y+\lambda_x\lambda_z+\lambda_y\lambda_z}{\lambda_x\lambda_y\lambda_z}\right)F\right](F+\lambda_x I)(F+\lambda_y I)(F+\lambda_z I) \tag{30}$$

3. $r(\lambda) = \lambda(\lambda+\lambda_y)(\lambda+\lambda_z)$

   $r'(\lambda) = 3\lambda^2 + 2(\lambda_y+\lambda_z)\lambda + \lambda_y\lambda_z$

   $F(F+\lambda_y I)(F+\lambda_z I) = \lambda_y\lambda_z Z_{12} + [-\lambda^3{}_x + (\lambda_y+\lambda_z)\lambda^2{}_x - \lambda_x\lambda_y\lambda_z]Z_{21}$

   $$Z_{21} = [-\lambda^3{}_x + (\lambda_y+\lambda_z)\lambda^2{}_x - \lambda_x\lambda_y\lambda_z]^{-1}F\left[I - \frac{1}{\lambda_x}(F+\lambda_x I)\right](F+\lambda_y I)(F+\lambda_z I) \tag{31}$$

4. $r(\lambda) = \lambda(\lambda+\lambda_x)(\lambda+\lambda_z)$

$r'(\lambda) = 3\lambda^2 + 2(\lambda_x+\lambda_z)\lambda+\lambda_x\lambda_z$

$F(F+\lambda_x I)(F+\lambda_z I) = \lambda_x\lambda_z Z_{12}+(-\lambda_y)[\lambda^2{}_y-(\lambda_x+\lambda_z)\lambda_y+\lambda_x\lambda_z]Z_{31}$

$$Z_{31} = [-\lambda^3{}_y+(\lambda_x+\lambda_z)\lambda^2{}_y+\lambda_x\lambda_y\lambda_z]^{-1}F\Big[(F+\lambda_x I)(F+\lambda_z I)- \left(\frac{1}{\lambda_y}\right)(F+\lambda_x I)(F+\lambda_y I)(F+\lambda_z I)\Big] \tag{32}$$

5. $r(\lambda) = \lambda(\lambda+\lambda_x)(\lambda+\lambda_y)$

$r'(\lambda) = 3\lambda^2 + 2(\lambda_x+\lambda_y)\lambda+\lambda_x\lambda_y$

$F(F+\lambda_x I)(F+\lambda_y I) = \lambda_x\lambda_y Z_{12} + (-\lambda_z)[\lambda^2{}_z-(\lambda_x+\lambda_y)\lambda_z+\lambda_x\lambda_y]Z_{41}$

$$Z_{41} = [-\lambda^3{}_z+(\lambda_x+\lambda_y)\lambda^2{}_z-\lambda_x\lambda_y\lambda_z]^{-1}F\Big[(F+\lambda_x I)(F+\lambda_y I) - \left(\frac{1}{\lambda_z}\right)(F+\lambda_x I)(F+\lambda_y I)(F+\lambda_z I)\Big]. \tag{33}$$

If Equations (29) through (33) are now substituted into Equation (28) the desired expression for e^{Ft} will be obtained as

$$\begin{aligned}
e^{Ft} = {} & \left(\frac{1}{\lambda_x\lambda_y\lambda_z}\right)(F+\lambda_x I)(F+\lambda_y I)(F+\lambda_z I) + \\
& \frac{e^{-\lambda_x t}}{[-\lambda^3{}_x+(\lambda_y+\lambda_z)\lambda^2{}_x-\lambda_x\lambda_y\lambda_z]} F(F+\lambda_y I)(F+\lambda_z I) + \\
& \frac{e^{-\lambda_y t}}{[-\lambda^3{}_y+(\lambda_x+\lambda_z)\lambda^2{}_y-\lambda_x\lambda_y\lambda_z]} F(F+\lambda_x I)(F+\lambda_z I) + \\
& \frac{e^{-\lambda_z t}}{[-\lambda^3{}_z+(\lambda_x+\lambda_y)\lambda^2{}_z-\lambda_x\lambda_y\lambda_z]} F(F+\lambda_x I)(F+\lambda_y I) - \\
& \Bigg[\frac{\lambda_x\lambda_y+\lambda_x\lambda_z+\lambda_y\lambda_z}{(\lambda_x\lambda_y\lambda_z)^2} - \frac{t}{\lambda_x\lambda_y\lambda_z} + \frac{e^{-\lambda_x t}}{[-\lambda^4{}_x+(\lambda_y+\lambda_z)\lambda^3{}_x-\lambda_y\lambda_z\lambda^2{}_x]} + \\
& \frac{e^{-\lambda_y t}}{[-\lambda^4{}_y+(\lambda_x+\lambda_z)\lambda^3{}_y-\lambda_x\lambda_z\lambda^2{}_y]} + \\
& \frac{e^{-\lambda_z t}}{[-\lambda^4{}_z+(\lambda_x+\lambda_y)\lambda^3{}_z-\lambda_x\lambda_y\lambda^2{}_z]}\Bigg] F(F+\lambda_x I)(F+\lambda_y I)(F+\lambda_z I).
\end{aligned} \tag{34}$$

Equation (34) seems formidable; however, if specific values are substituted for λ_x, λ_y, λ_z the coefficient terms simplify. Equation (34) mainly involves several combinations of matrix multiplication. However, for the general case indicated in Equation (34), when the matrix multiplications are performed and the terms are combined as indicated, the resulting exact expression for the state transition matrix (for the λ's of this section) is as shown in Equation (35).

$$
e^{Ft} = \left[\begin{array}{ccc|ccc|ccc}
1 & 0 & 0 & t & 0 & 0 & \frac{t}{\lambda_x} - \frac{1}{\lambda_x^2} + \frac{(-\lambda_x^2+\lambda_x\lambda_y+\lambda_x\lambda_z-\lambda_y\lambda_z)e^{-\lambda_x t}}{[-\lambda_x^4+(\lambda_y+\lambda_z)\lambda_x^3-\lambda_x^2\lambda_y\lambda_z]} & 0 & 0 \\
0 & 1 & 0 & 0 & t & 0 & 0 & \frac{t}{\lambda_y} - \frac{1}{\lambda_y^2} + \frac{(-\lambda_y^2+\lambda_x\lambda_y+\lambda_z\lambda_y-\lambda_x\lambda_z)e^{-\lambda_y t}}{[-\lambda_y^4+(\lambda_x+\lambda_z)\lambda_y^3-\lambda_x\lambda_y^2\lambda_z]} & 0 \\
0 & 0 & 1 & 0 & t & 0 & 0 & 0 & \frac{t}{\lambda_z} - \frac{1}{\lambda_z^2} + \frac{(-\lambda_z^2+\lambda_z\lambda_x+\lambda_z\lambda_y-\lambda_x\lambda_y)e^{-\lambda_z t}}{[-\lambda_z^4+(\lambda_x+\lambda_y)\lambda_z^3-\lambda_x\lambda_y\lambda_z]} \\
\hline
0 & 0 & 0 & 1 & 0 & 0 & \frac{1}{\lambda_x} - \frac{(-\lambda_x^2+\lambda_x\lambda_z+\lambda_x\lambda_y-\lambda_y\lambda_z)e^{-\lambda_x t}}{[-\lambda_x^3+(\lambda_y+\lambda_z)\lambda_x^2-\lambda_x\lambda_y\lambda_z]} & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & \frac{1}{\lambda_y} - \frac{(-\lambda_y^2+\lambda_y\lambda_z+\lambda_y\lambda_x-\lambda_x\lambda_z)e^{-\lambda_y t}}{[-\lambda_y^3+(\lambda_x+\lambda_z)\lambda_y^2-\lambda_x\lambda_y\lambda_z]} & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \frac{1}{\lambda_z} - \frac{(-\lambda_z^2+\lambda_z\lambda_x+\lambda_z\lambda_y-\lambda_x\lambda_y)e^{-\lambda_z t}}{[-\lambda_z^3+(\lambda_x+\lambda_y)\lambda_z^2-\lambda_x\lambda_y\lambda_z]} \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & \frac{\lambda_x(-\lambda_x^2+\lambda_x\lambda_y+\lambda_x\lambda_z-\lambda_y\lambda_z)e^{-\lambda_x t}}{[-\lambda_x^3+(\lambda_y+\lambda_z)\lambda_x^2-\lambda_x\lambda_y\lambda_z]} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\lambda_y(-\lambda_y^2-\lambda_x\lambda_z+\lambda_x\lambda_y+\lambda_y\lambda_z)e^{-\lambda_y t}}{[-\lambda_y^3+(\lambda_x+\lambda_z)\lambda_y^2-\lambda_x\lambda_y\lambda_z]} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\lambda_z(-\lambda_z^2-\lambda_x\lambda_y+\lambda_x\lambda_z+\lambda_y\lambda_z)e^{-\lambda_z t}}{[-\lambda_z^3+(\lambda_x+\lambda_y)\lambda_z^2-\lambda_x\lambda_y\lambda_z]}
\end{array}\right] \tag{35}
$$

C. Case With λ_x, λ_y, λ_z Positive, Equal

For the case where $\lambda_x = \lambda_y = \lambda_z$, the minimal polynomial of F is found to be

$$\Psi(\lambda) = \lambda^2 \; (\lambda+\lambda_x) \tag{36}$$

and the fundamental formula for f(F), as given by Equation (12), is as follows,

$$\begin{aligned} f(F) &= f(\lambda_1)Z_{11} + f'(\lambda_1)Z_{12} + f(\lambda_2)Z_{21} \\ &= f(0)Z_{11} + f'(0)Z_{12} + f(-\lambda_x)Z_{21}. \end{aligned} \tag{37}$$

If one again chooses $f(\lambda) = e^{\lambda t}$ then one has $f(F) = e^{Ft}$ so that, using Equation (37)

$$e^{Ft} = Z_{11} + tZ_{12} + e^{-\lambda_x t}Z_{21}. \tag{38}$$

The components of F can be found by using the same procedure as in paragraph B, with $r(\lambda)$ given as $r(\lambda) = r(\lambda_1)Z_{11} + r'(\lambda_1)Z_{12} + r(\lambda_2)Z_{21}$. Proceeding as in paragraph B above, one has the following results.

1. $r(\lambda) = 1$

   $r'(\lambda) = 0$

   $$I = Z_{11} + Z_{21} \tag{39}$$

2. $r(\lambda) = \lambda(\lambda+\lambda_x)$

   $r'(\lambda) = 2\lambda + \lambda_x$

   $$F(F+\lambda_x I) = Z_{12} \tag{40}$$

3. $r(\lambda) = \lambda^2$

   $r'(\lambda) = 2\lambda$

   $$F^2 = Z_{21} \tag{41}$$

From steps (1) and (3), Z_{11} may be found to be

$$Z_{11} = I - F^2. \tag{42}$$

If Equations (40) through (42) are substituted into Equation (38), the expression for e^{Ft} will be obtained as

$$e^{Ft} = I - F^2 + F^2 t + Ft + e^{-\lambda_x t} F^2$$

$$= I + (e^{-\lambda_x t} + t - 1) F^2 + Ft \qquad (43)$$

When Equation (43) is expanded, the resultant matrix expression for e^{Ft} (for the λ's of this paragraph) is as follows,

$$e^{Ft} = \left[\begin{array}{ccc|ccc|ccc}
1 & 0 & 0 & t & 0 & 0 & a & 0 & 0 \\
0 & 1 & 0 & 0 & t & 0 & 0 & a & 0 \\
0 & 0 & 1 & 0 & 0 & t & 0 & 0 & a \\
\hline
 & & & 1 & 0 & 0 & b & 0 & 0 \\
 & 0 & & 0 & 1 & 0 & 0 & b & 0 \\
 & & & 0 & 0 & 1 & 0 & 0 & b \\
\hline
 & & & & & & c & 0 & 0 \\
 & 0 & & & 0 & & 0 & c & 0 \\
 & & & & & & 0 & 0 & c
\end{array}\right] \qquad (44)$$

where

$$a = t - 1 + e^{-\lambda_x t} \qquad (45)$$

$$b = (1-\lambda_x)t + (1-e^{-\lambda_x t})\lambda_x \qquad (46)$$

$$c = 1 + (e^{-\lambda_x t}-1)\lambda_x{}^2 + (\lambda_x-1)\lambda_x t. \qquad (47)$$

D. Case With One Root Zero; Remaining Roots Positive, Unequal

To begin this paragraph, assume that $\lambda_y = 0$ and that λ_x and λ_z are positive, unequal. Based on this assumption the minimal polynomial of F is found to be

$$\Psi(\lambda) = \lambda^3 (\lambda+\lambda_x)(\lambda+\lambda_z). \qquad (48)$$

For the conditions of this paragraph, the fundamental formula for f(F) is

$$f(F) = f(-\lambda_x)Z_{11} + f(0)Z_{21} + f'(0)Z_{22} + f''(0)Z_{23} + f(-\lambda_z)Z_{31} \qquad (49)$$

and e^{Ft} will be given by

$$e^{Ft} = e^{-\lambda_x t}Z_{11} + Z_{21} + tZ_{22} + t^2 Z_{23} + e^{-\lambda_z t}Z_{31}. \qquad (50)$$

The components of F are again found by the same procedure used in paragraphs B and C where, in this paragraph, $r(\lambda)$ is given by $r(\lambda) = r(-\lambda_x)Z_{11} + r(0)Z_{21} + r'(0)Z_{22} + r''(0)Z_{23} + r(-\lambda_z)Z_{31}$.

1. $r(\lambda) = \lambda^3(\lambda+\lambda_x)$

$$r'(\lambda) = 4\lambda^3 + 3\lambda_x\lambda^2$$

$$r''(\lambda) = 12\lambda^2 + 6\lambda_x\lambda$$

$$F^3(F+\lambda_x I) = -\lambda^3{}_z(\lambda_x-\lambda_z)Z_{31}$$

$$Z_{31} = -\frac{1}{\lambda^3{}_z(\lambda_x-\lambda_z)}(F^4+\lambda_x F^3) \tag{51}$$

2. $r(\lambda) = \lambda^3$

$$r'(\lambda) = 3\lambda^2$$

$$r''(\lambda) = 6\lambda$$

$$F^3 = -\lambda^3{}_x Z_{11} - \lambda^3{}_z Z_{31}$$

$$Z_{11} = -\left(\frac{1}{\lambda^3{}_x}\right)F^3 + \frac{1}{\lambda^3{}_x(\lambda_x-\lambda_z)}(F^4+\lambda_x F^3) \tag{52}$$

3. $r(\lambda) = 1$

$$r'(\lambda) = r''(\lambda) = 0$$

$$I = Z_{11} + Z_{21} + Z_{31}$$

$$Z_{21} = I - Z_{11} - Z_{31} = I + \frac{F^3}{\lambda^3{}_x} + \left(\frac{\lambda^3{}_x-\lambda^3{}_z}{\lambda^3{}_x\lambda^3{}_z}\right)\left(\frac{F^4+\lambda_x F^3}{\lambda_x-\lambda_z}\right) \tag{53}$$

4. $r(\lambda) = \lambda^2$

$$r'(\lambda) = 2\lambda$$

$$r''(\lambda) = 2$$

$$F^2 = \lambda^2{}_x Z_{11} + 2Z_{23} + \lambda^2{}_z Z_{31}$$

$$Z_{23} = 1/2(F^2-\lambda^2{}_x Z_{11}-\lambda^2{}_z Z_{31})$$

$$= 1/2\left[F^2 + \left(\frac{\lambda_x+\lambda_z}{\lambda_x\lambda_z}\right)F^3 + \frac{1}{\lambda_x\lambda_z}F^4\right]. \tag{54}$$

5. $r(\lambda) = \lambda(\lambda+\lambda_x)(\lambda+\lambda_z)$

$$r'(\lambda) = 3\lambda^2 + 2(\lambda_x+\lambda_z)\lambda + \lambda_x\lambda_z$$

$$r''(\lambda) = 6\lambda + 2(\lambda_x+\lambda_z)$$

$$F(F+\lambda_x I)(F+\lambda_z I) = \lambda_x\lambda_z Z_{22} + 2(\lambda_x+\lambda_z)Z_{23}$$

$$Z_{22} = -\left(\frac{\lambda_x+\lambda_z}{\lambda^2{}_x\lambda^2{}_z}\right)F^4 - \left(\frac{\lambda^2{}_x+\lambda_x\lambda_z+\lambda^2{}_z}{\lambda^2{}_x\lambda^2{}_z}\right)F^3 + F. \tag{55}$$

If Equations (51) through (55) are substituted into Equation (50), the desired expression for e^{Ft} (for the λ's of this paragraph) is given in matrix form as shown in Equation (56).

$$e^{Ft} = \left[\begin{array}{ccc|ccc|ccc}
1 & 0 & 0 & t & 0 & 0 & a\lambda^2_x-b\lambda_x+\tfrac{1}{2}t^2 & 0 & 0 \\
0 & 1 & 0 & 0 & t & 0 & 0 & \tfrac{1}{2}t^2 & 0 \\
0 & 0 & 1 & 0 & 0 & t & 0 & 0 & a\lambda^2_z-b\lambda_z+\tfrac{1}{2}t^2 \\
\hline
 & & & 1 & 0 & 0 & -a\lambda^3_x+b\lambda^2_x-\tfrac{1}{2}\lambda_x t^2+t & 0 & 0 \\
 & 0 & & 0 & 1 & 0 & 0 & t & 0 \\
 & & & 0 & 0 & 1 & 0 & 0 & -a\lambda^3_z+b\lambda^2_z-\tfrac{1}{2}\lambda_z t^2+t \\
\hline
 & & & & & & a\lambda^4_x-b\lambda^3_x+\tfrac{1}{2}\lambda^2_x t^2-\lambda_x t+1 & 0 & 0 \\
 & 0 & & & 0 & & 0 & 1 & 0 \\
 & & & & & & 0 & 0 & a\lambda^4_z-b\lambda^3_z+\tfrac{1}{2}\lambda^2_z t^2-\lambda_z t+1
\end{array}\right] \tag{56}$$

where

$$a = \frac{e^{-\lambda_x t}}{\lambda^3_x(\lambda_x-\lambda_z)} + \left(\frac{\lambda^2_x+\lambda_x\lambda_z+\lambda^2_z}{\lambda^3_x\lambda^3_z}\right) - \left(\frac{\lambda_x+\lambda_z}{\lambda^2_x\lambda^2_z}\right)t + \frac{t^2}{2\lambda_x\lambda_z} - \frac{e^{-\lambda_z t}}{\lambda^3_z(\lambda_x-\lambda_z)}$$

$$b = \frac{\lambda_z e^{-\lambda_x t}}{\lambda^3_x(\lambda_x-\lambda_z)} + \left(\frac{\lambda^3_x+\lambda^2_x\lambda_z+\lambda_x\lambda^2_z+\lambda^3_z}{\lambda^3_x\lambda^3_z}\right) - \left(\frac{\lambda^2_x+\lambda_x\lambda_z+\lambda^2_z}{\lambda^2_x\lambda^2_z}\right)t + \left(\frac{\lambda_x+\lambda_z}{2\lambda_x\lambda_z}\right)t^2 - \frac{\lambda_x e^{-\lambda_z t}}{\lambda^3_z(\lambda_x-\lambda_z)} .$$

For a case where $\lambda_x = 0$, with λ_y, λ_z positive, unequal, the minimal polynomial of F is

$$\Psi(F) = \lambda^3(\lambda+\lambda_y)(\lambda+\lambda_z) \tag{57}$$

and the fundamental formula for f(F) is given by

$$f(F) = f(0)Z_{11} + f'(0)Z_{12} + f''(0)Z_{13} + f(-\lambda_y)Z_{21} + f(-\lambda_z)Z_{31} . \tag{58}$$

In a similar fashion, for $\lambda_z = 0$ and λ_x, λ_y positive, unequal, the minimal polynomial of F is

$$\Psi(\lambda) = \lambda^3(\lambda+\lambda_x)(\lambda+\lambda_y) \tag{59}$$

and the fundamental formula for f(F) is given by

$$f(F) = f(-\lambda_x)Z_{11} + f(-\lambda_y)Z_{21} + f(0)Z_{31} + f'(0)Z_{32} + f''(0)Z_{33} . \tag{60}$$

The expressions for e^{Ft} corresponding to Equations (58) and (60) are given in Equations (61) and (62), respectively.

$$
e^{Ft} = \left[\begin{array}{ccc|ccc|ccc}
1 & 0 & 0 & t & 0 & 0 & \tfrac{1}{2}t^2 & 0 & 0 \\
0 & 1 & 0 & 0 & t & 0 & 0 & a\lambda^2_y - b\lambda_y + \tfrac{1}{2}t^2 & 0 \\
0 & 0 & 1 & 0 & 0 & t & 0 & 0 & a\lambda^2_z - b\lambda_z + \tfrac{1}{2}t^2 \\
\hline
 & & & 1 & 0 & 0 & t & 0 & 0 \\
 & 0 & & 0 & 1 & 0 & 0 & -a\lambda^3_y + b\lambda^2_y - \tfrac{1}{2}t^2\lambda_y + t & 0 \\
 & & & 0 & 0 & 1 & 0 & 0 & -a\lambda^3_z + b\lambda^2_z - \tfrac{1}{2}t^2\lambda_z + t \\
\hline
 & & & & & & 1 & 0 & 0 \\
 & 0 & & & 0 & & 0 & a\lambda^4_y - b\lambda^3_y + \frac{t^2}{2}\lambda^2_y - \lambda_y t + 1 & 0 \\
 & & & & & & 0 & 0 & a\lambda^4_z - b\lambda^3_z + \frac{t^2}{2}\lambda^2_z - \lambda_z t + 1
\end{array}\right] \tag{61}
$$

where

$$
a = \left(\frac{1}{\lambda_y - \lambda_z}\right)\left(\frac{e^{-\lambda_y t}}{\lambda^3_y} - \frac{e^{-\lambda_z t}}{\lambda^3_z}\right) + \frac{t^2}{2\lambda_y\lambda_z} - \left(\frac{\lambda_y + \lambda_z}{\lambda^2_y\lambda^2_z}\right)t + \left(\frac{\lambda^2_y + \lambda_y\lambda_z + \lambda^2_z}{\lambda^3_y\lambda^3_z}\right),
$$

$$
b = \frac{\lambda_z e^{-\lambda_y t}}{\lambda^3_y(\lambda_y - \lambda_z)} - \frac{\lambda_y e^{-\lambda_z t}}{\lambda^3_z(\lambda_y - \lambda_z)} + \left(\frac{\lambda_y + \lambda_z}{2\lambda_y\lambda_z}\right)t^2 - \left(\frac{\lambda^2_y + \lambda_y\lambda_z + \lambda^2_z}{\lambda^2_y\lambda^2_z}\right)t + \left(\frac{\lambda^3_y + \lambda^2_y\lambda_z + \lambda_y\lambda^2_z + \lambda^3_z}{\lambda^3_y\lambda^3_z}\right).
$$

$$
e^{Ft} = \left[\begin{array}{ccc|ccc|ccc}
1 & 0 & 0 & t & 0 & 0 & a\lambda^2_x - b\lambda_x + \tfrac{1}{2}t^2 & 0 & 0 \\
0 & 1 & 0 & 0 & t & 0 & 0 & a\lambda^2_y - b\lambda_y + \tfrac{1}{2}t^2 & 0 \\
0 & 0 & 1 & 0 & 0 & t & 0 & 0 & \tfrac{1}{2}t^2 \\
\hline
 & & & 1 & 0 & 0 & -a\lambda^3_x + b\lambda^2_x - \tfrac{1}{2}\lambda_x t^2 + t & 0 & 0 \\
 & 0 & & 0 & 1 & 0 & 0 & -a\lambda^3_y + b\lambda^2_y - \tfrac{1}{2}\lambda_y t^2 + t & 0 \\
 & & & 0 & 0 & 1 & 0 & 0 & t \\
\hline
 & & & & & & a\lambda^4_x - b\lambda^3_x + \frac{t^2}{2}\lambda_x - \lambda_x t + 1 & 0 & 0 \\
 & 0 & & & 0 & & 0 & a\lambda^4_y - b\lambda^3_y + \frac{t^2}{2}\lambda^2_y - \lambda_y t + 1 & 0 \\
 & & & & & & 0 & 0 & 1
\end{array}\right] \tag{62}
$$

where

$$a = \left(\frac{1}{\lambda_x-\lambda_y}\right)\left(\frac{e^{-\lambda_x t}}{\lambda^3{}_x} - \frac{e^{-\lambda_y t}}{\lambda^3{}_y}\right) + \frac{t^2}{2\lambda_x\lambda_y} - \left(\frac{\lambda_x+\lambda_y}{\lambda^2{}_x\lambda^2{}_y}\right)t + \left(\frac{\lambda^2{}_x+\lambda_x\lambda_y+\lambda^2{}_y}{\lambda^3{}_x\lambda^3{}_y}\right)$$

$$b = \frac{\lambda_y e^{-\lambda_x t}}{\lambda^3{}_x(\lambda_x-\lambda_y)} - \frac{\lambda_x e^{-\lambda_y t}}{\lambda^3{}_y(\lambda_x-\lambda_y)} + \left(\frac{\lambda_x+\lambda_y}{2\lambda_x\lambda_y}\right)t^2 - \left(\frac{\lambda^2{}_x+\lambda_x\lambda_y+\lambda^2{}_y}{\lambda^2{}_x\lambda^2{}_y}\right)t + \left(\frac{\lambda^3{}_x+\lambda^2{}_x\lambda_y+\lambda_x\lambda^2{}_y+\lambda^3{}_y}{\lambda^3{}_x\lambda^3{}_y}\right)$$

E. Case With One Root Zero; Remaining Roots Positive, Equal

For a case with, for instance, $\lambda_x = 0$, $\lambda_y = \lambda_z > 0$, the minimal polynomial of F is found to be

$$\Psi(\lambda) = \lambda^3(\lambda+\lambda_y) \tag{63}$$

and the fundamental formula for f(F) is thus given by

$$f(F) = f(0)Z_{11} + f'(0)Z_{12} + f''(0)Z_{13} + f(-\lambda_y)Z_{21}. \tag{64}$$

The expression for e^{Ft} is, therefore, given as

$$e^{Ft} = Z_{11} + tZ_{12} + t^2Z_{13} + e^{-\lambda_y t}Z_{21} \tag{65}$$

and the components of F can be found as follows:

1. $r(\lambda) = \lambda^3$

   $r'(\lambda) = 3\lambda^2$; $r''(\lambda) = 6\lambda$

   $F^3 = -\lambda^3{}_y Z_{21}$

   $$Z_{21} = -\left(\frac{1}{\lambda^3{}_y}\right)F^3 \tag{66}$$

2. $r(\lambda) = \lambda^2(\lambda+\lambda_y)$

   $r'(\lambda) = 3\lambda^2 + 2\lambda_y\lambda$; $r''(\lambda) = 6\lambda + 2\lambda_y$

   $F^2(F+\lambda_y I) = 2\lambda_y Z_{13}$

   $$Z_{13} = \left(\frac{1}{2\lambda_y}\right)F^2(F+\lambda_y I) \tag{67}$$

3. $r(\lambda) = 1$

   $r'(\lambda) = r''(\lambda) = 0$

   $I = Z_{11} + Z_{21}$

$$Z_{11} = I + \left(\frac{1}{\lambda^3_y}\right)F^3 \tag{68}$$

4. $r(\lambda) = \lambda(\lambda+\lambda_y)$

$$r'(\lambda) = 2\lambda + \lambda_y \; ; \; r''(\lambda) = 2$$

$$F(F+\lambda_y I) = \lambda_y Z_{12} + 2Z_{13}$$

$$Z_{12} = \left(\frac{1}{\lambda_y}\right)F(F+\lambda_y I) - \left(\frac{1}{\lambda^2_y}\right)F^2(F+\lambda_y I). \tag{69}$$

If one substitutes Equations (66) through (69) into Equation (65), the desired matrix expression for e^{Ft} can be written as shown in Equation (70).

$$e^{Ft} = \left[\begin{array}{ccc|ccc|ccc}
1 & 0 & 0 & t & 0 & 0 & \frac{1}{2}t^2 & 0 & 0 \\
0 & 1 & 0 & 0 & t & 0 & 0 & \frac{1}{2}t^2-a\lambda_y & 0 \\
0 & 0 & 1 & 0 & 0 & t & 0 & 0 & \frac{1}{2}t^2-a\lambda_y \\
\hline
 & & & 1 & 0 & 0 & t & 0 & 0 \\
 & 0 & & 0 & 1 & 0 & 0 & t-\frac{1}{2}t^2\lambda_y+a\lambda^2_y & 0 \\
 & & & 0 & 0 & 1 & 0 & 0 & t-\frac{1}{2}t^2\lambda_y+a\lambda^2_y \\
\hline
 & & & & & & 1 & 0 & 0 \\
 & 0 & & & 0 & & 0 & 1-\lambda_y t+\frac{t^2}{2}\lambda^2_y-a\lambda^3_y & 0 \\
 & & & & & & 0 & 0 & 1-\lambda_y t+\frac{t^2}{2}\lambda^2_y-a\lambda^3_y
\end{array}\right] \tag{70}$$

where

$$a = \frac{1}{\lambda^3_y} - \frac{t}{\lambda^2_y} + \frac{t^2}{2\lambda_y} - \frac{e^{-\lambda_y t}}{\lambda^3_y} .$$

If one has a case with $\lambda_z = 0$, $\lambda_x = \lambda_y > 0$, then e^{Ft} is as shown in Equation (71).

$$
e^{Ft} = \left[\begin{array}{ccc|ccc|ccc}
1 & 0 & 0 & t & 0 & 0 & \frac{t^2}{2} - a\lambda_x & 0 & 0 \\
0 & 1 & 0 & 0 & t & 0 & 0 & \frac{t^2}{2} - a\lambda_x & 0 \\
0 & 0 & 1 & 0 & 0 & t & 0 & 0 & \frac{t^2}{2} \\
\hline
 & & & 1 & 0 & 0 & t - \frac{t^2}{2}\lambda_x + a\lambda^2_x & 0 & 0 \\
 & 0 & & 0 & 1 & 0 & 0 & t - \frac{t^2}{2}\lambda_x + a\lambda^2_x & 0 \\
 & & & 0 & 0 & 1 & 0 & 0 & t \\
\hline
 & & & & & & 1 - \lambda_x t + \frac{t^2}{2}\lambda^2_x - a\lambda^3_x & 0 & 0 \\
 & 0 & & & 0 & & 0 & 1 - \lambda_x t + \frac{t^2}{2}\lambda^2_x - a\lambda^3_x & 0 \\
 & & & & & & 0 & 0 & 1
\end{array}\right] \tag{71}
$$

where

$$
a = \frac{1}{\lambda^3_y} - \frac{t}{\lambda^2_y} + \frac{t^2}{2\lambda_y} - \frac{e^{-\lambda_y t}}{\lambda^3_y} .
$$

For a case with $\lambda_y = 0$, $\lambda_x = \lambda_z > 0$, e^{Ft} will be as shown in Equation (72).

$$
e^{Ft} = \left[\begin{array}{ccc|ccc|ccc}
1 & 0 & 0 & t & 0 & 0 & \frac{t^2}{2} - a\lambda_x & 0 & 0 \\
0 & 1 & 0 & 0 & t & 0 & 0 & \frac{t^2}{2} & 0 \\
0 & 0 & 1 & 0 & 0 & t & 0 & 0 & \frac{t^2}{2} - a\lambda_x \\
\hline
 & & & 1 & 0 & 0 & t - \frac{t^2}{2}\lambda_x + a\lambda^2_x & 0 & 0 \\
 & 0 & & 0 & 1 & 0 & 0 & t & 0 \\
 & & & 0 & 0 & 1 & 0 & 0 & t - \frac{t^2}{2}\lambda_x + a\lambda^2_x \\
\hline
 & & & & & & 1 - \lambda_x t + \frac{t^2}{2}\lambda^2_x - a\lambda^3_x & 0 & 0 \\
 & 0 & & & 0 & & 0 & 1 & 0 \\
 & & & & & & 0 & 0 & 1 - \lambda_x t + \frac{t^2}{2}\lambda^2_x - a\lambda^3_x
\end{array}\right] \tag{72}
$$

where

$$a = \frac{1}{\lambda^3_x} - \frac{t}{\lambda^2_x} + \frac{t^2}{2\lambda_x} - \frac{e^{-\lambda_x t}}{\lambda^3_x} .$$

IV. CONCLUSIONS

In order to verify the expressions developed for e^{Ft} in Section III, a computer program was written which would integrate the target equations (2), for an arbitrary set of initial conditions and λ's and for $w \equiv 0$, to generate time histories of target position, velocity and acceleration along the x, y, z axes. A listing of the program is presented in Appendix A. Using this program, data was generated for each of the cases considered in Section III. Table 1 lists the target state initial conditions and the values used for λ_x, λ_y and λ_z for each computer run case. The target state transition matrix from Section III, which corresponds to each of the cases listed in Table 1, was then used to generate target state data for comparison with the computer output.

A representative set of data indicating the results of the comparison is presented in Figures 1 through 12. This data represents the target position, for each of the cases in Table 1, for both the computer solution and the state transition matrix solution. As can be seen, the data output matches in each instance and the state transition matrix expressions in Section III are verified.

TABLE 1. TEST RUN CONDITIONS

Case	λ_x	λ_y	λ_z	Paragraph
1	1.	1.	1.	C
2	3.	2.	1.	B
3	2.	0.	1.	D
6	0.	1.	1.	E

NOTE: Target state initial conditions in each case were as follows:

$x(0) = 100.$ $\quad y(0) = 200.$ $\quad z(0) = 2000.$

$\dot{x}(0) = 2000.$ $\quad \dot{y}(0) = 400.$ $\quad \dot{z}(0) = 300.$

$\ddot{x}(0) = 300.$ $\quad \ddot{y}(0) = -2000.$ $\quad \ddot{z}(0) = 700.$

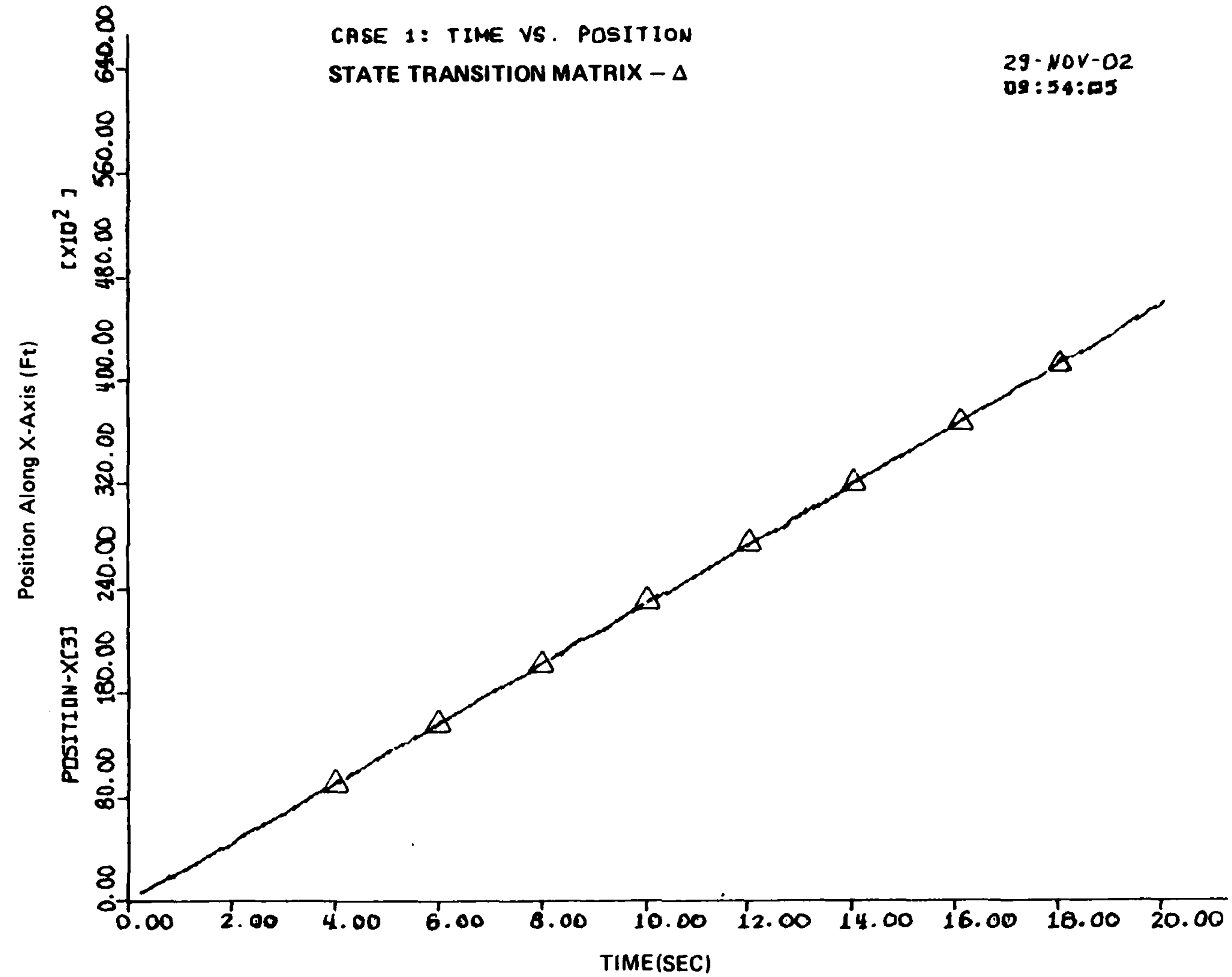

Figure 1. Target x Position Versus Time, Case 1.

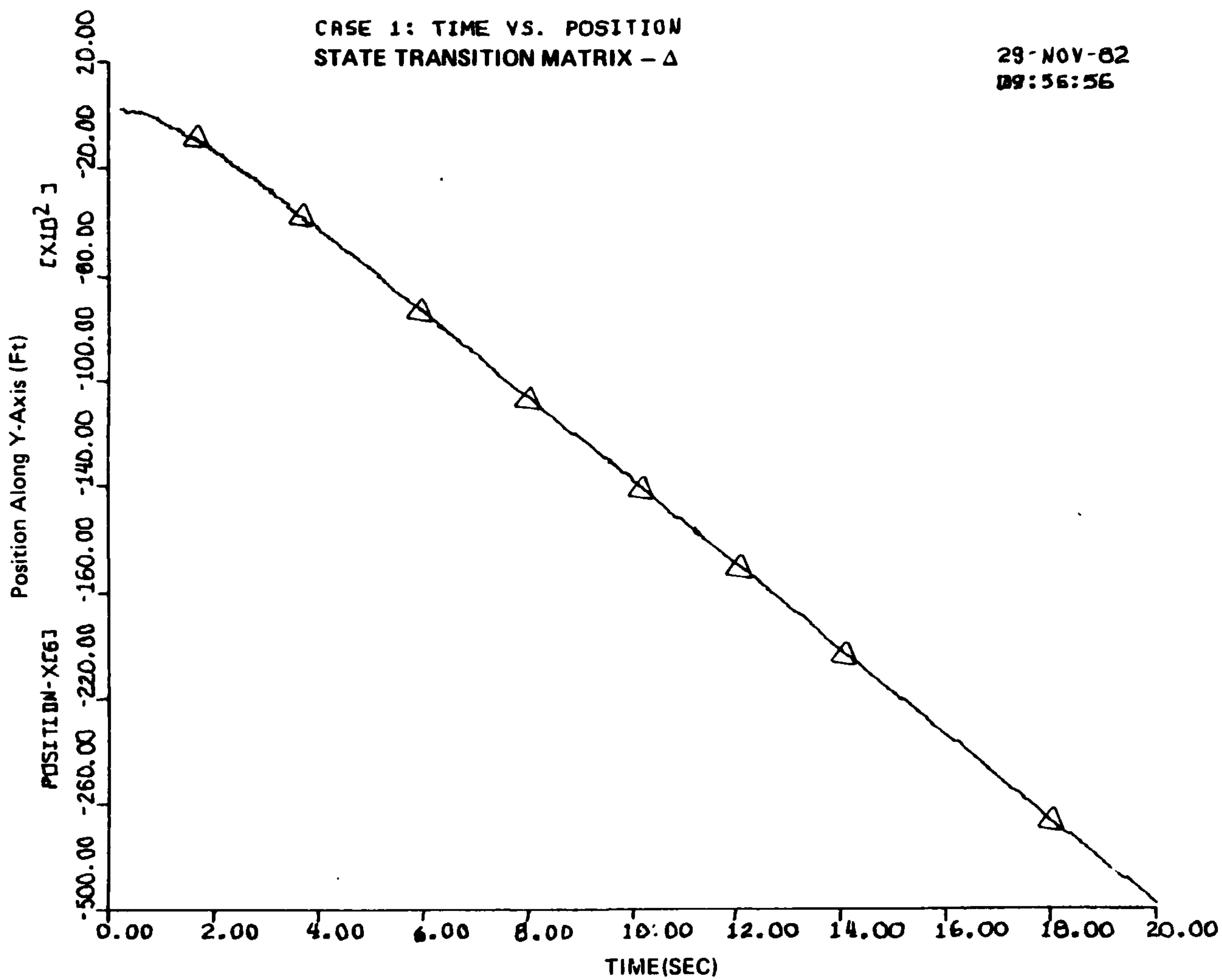

Figure 2. Target y Position Versus Time, Case 1.

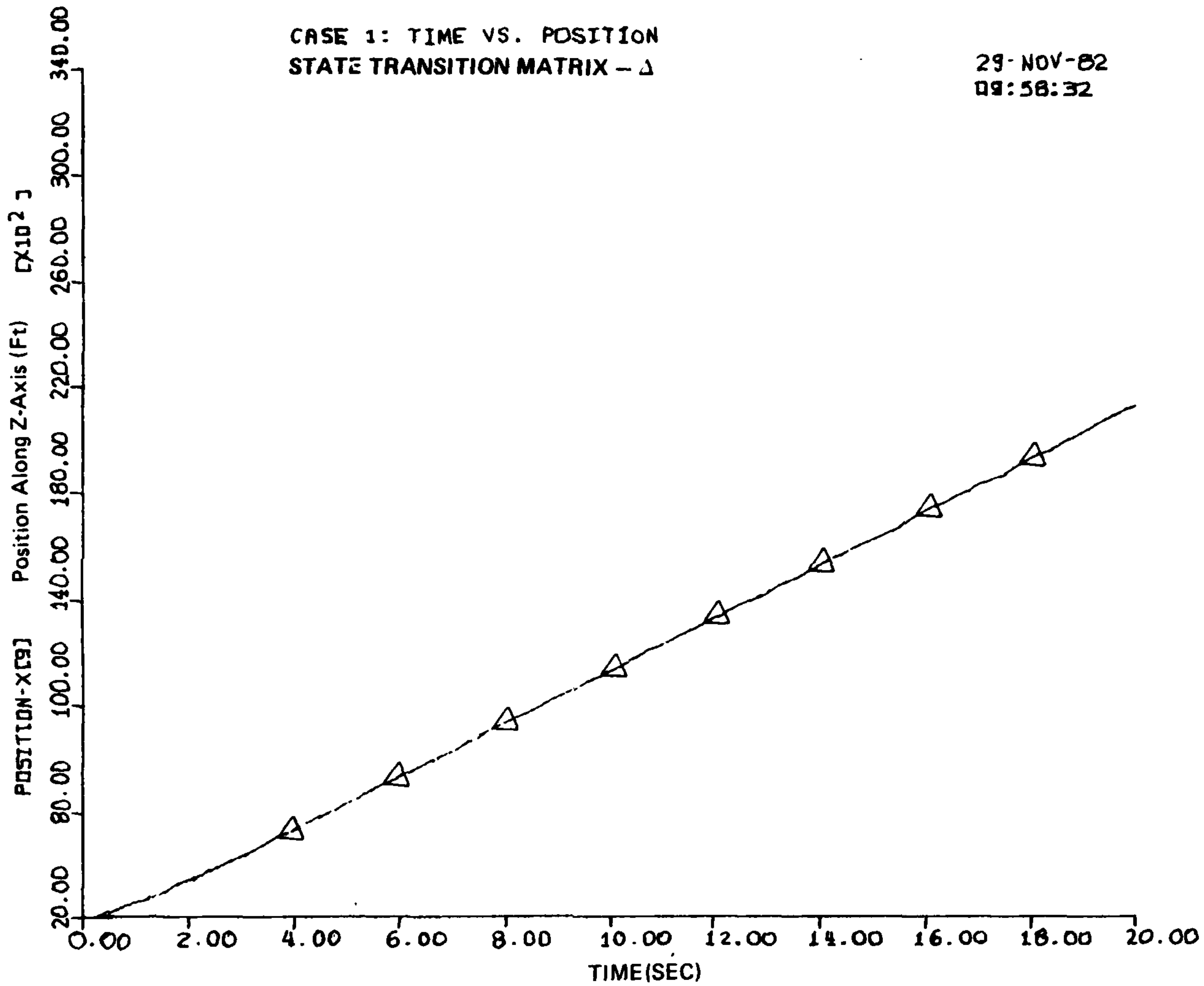

Figure 3. Target z Position Versus Time, Case 1.

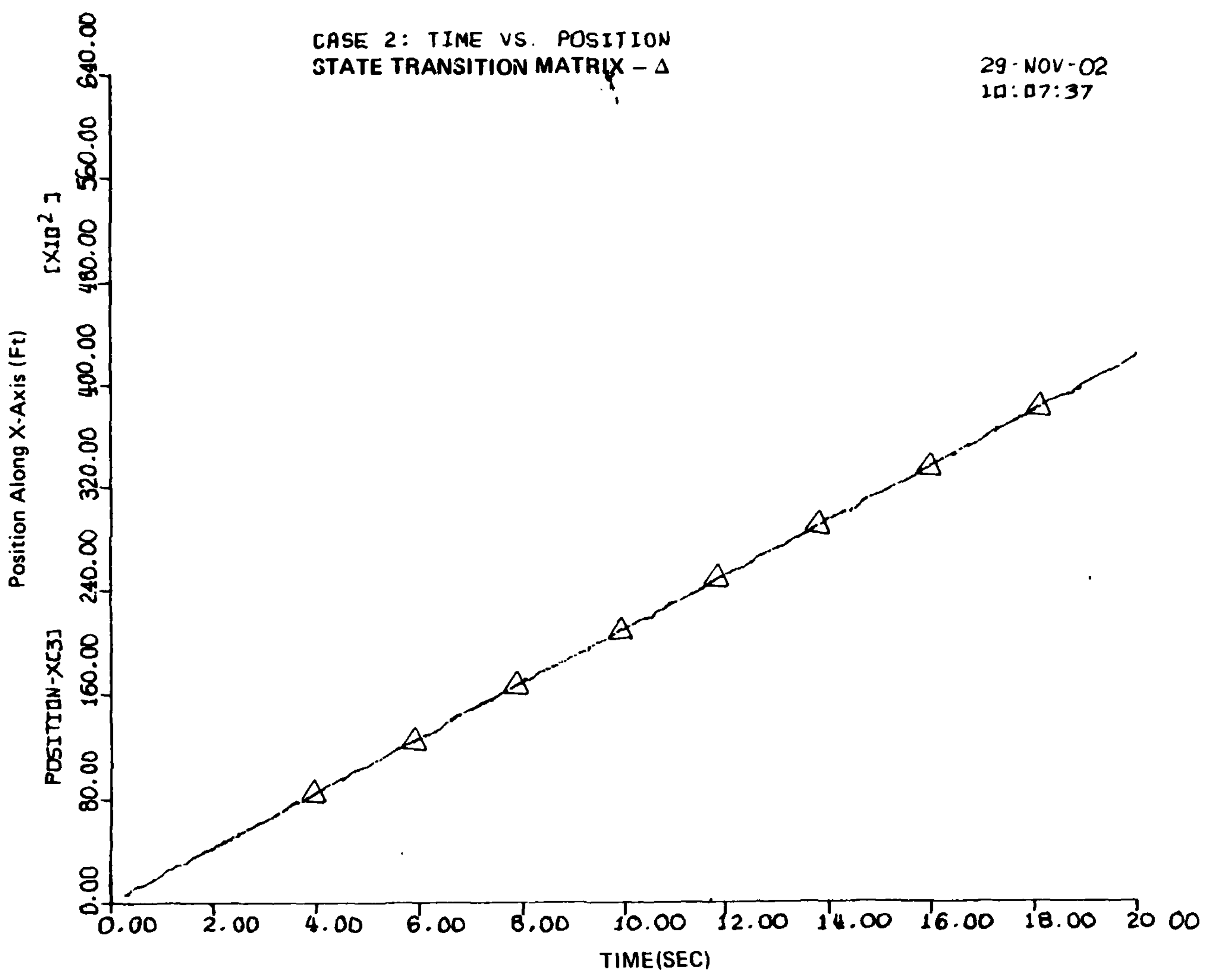

Figure 4. Target x Position Versus Time, Case 2.

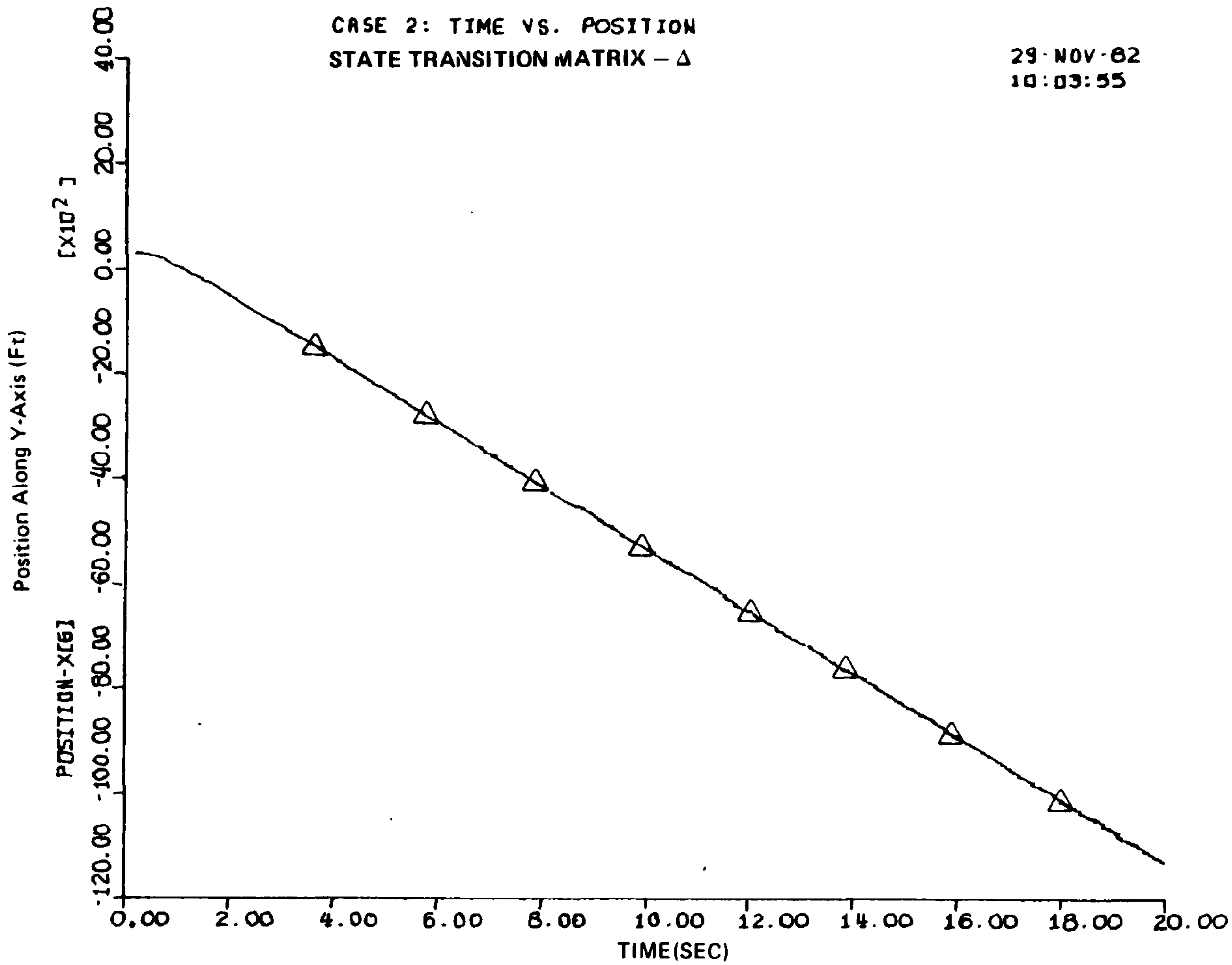

Figure 5. Target y Position Versus Time, Case 2.

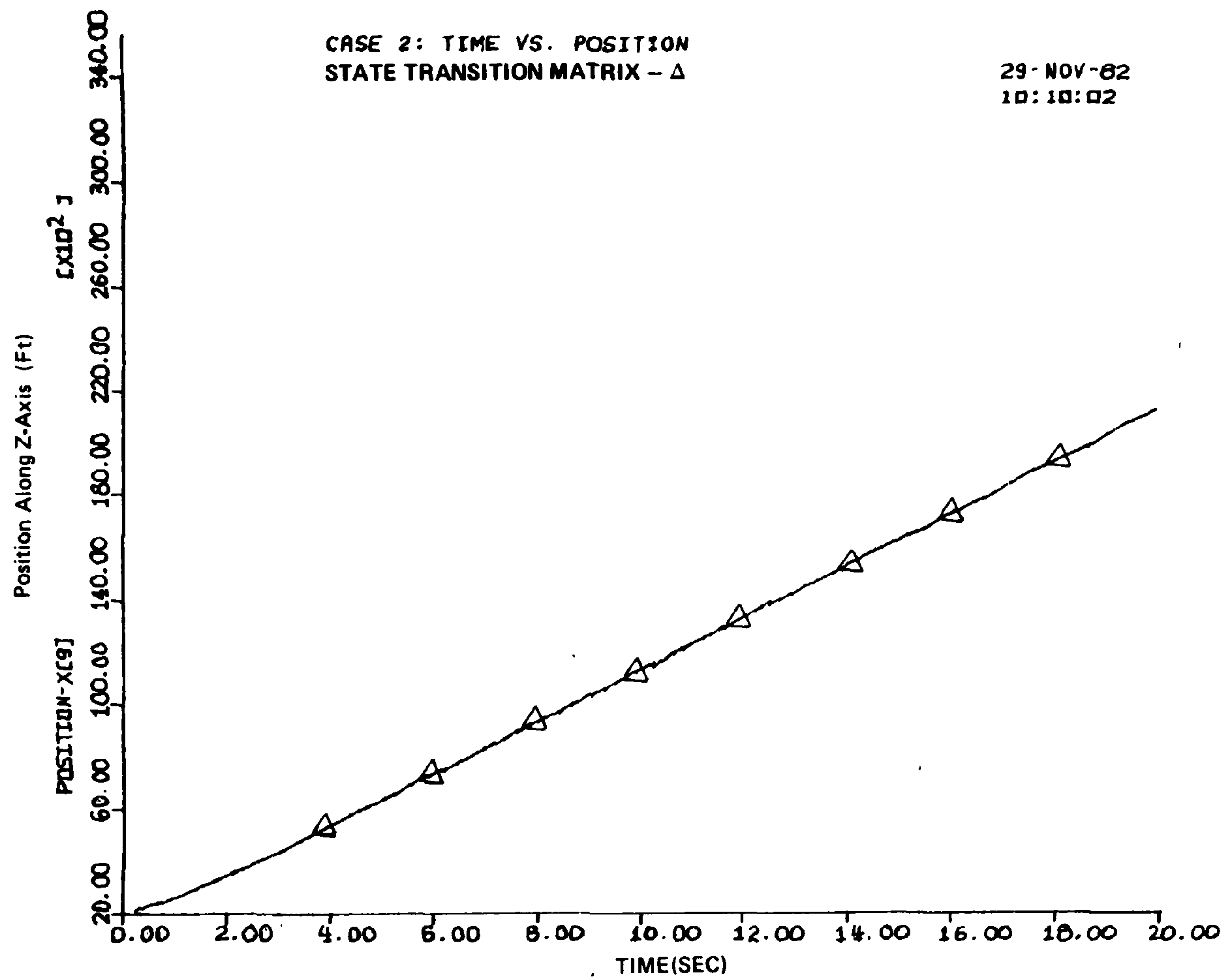

Figure 6. Target z Position Versus Time, Case 2.

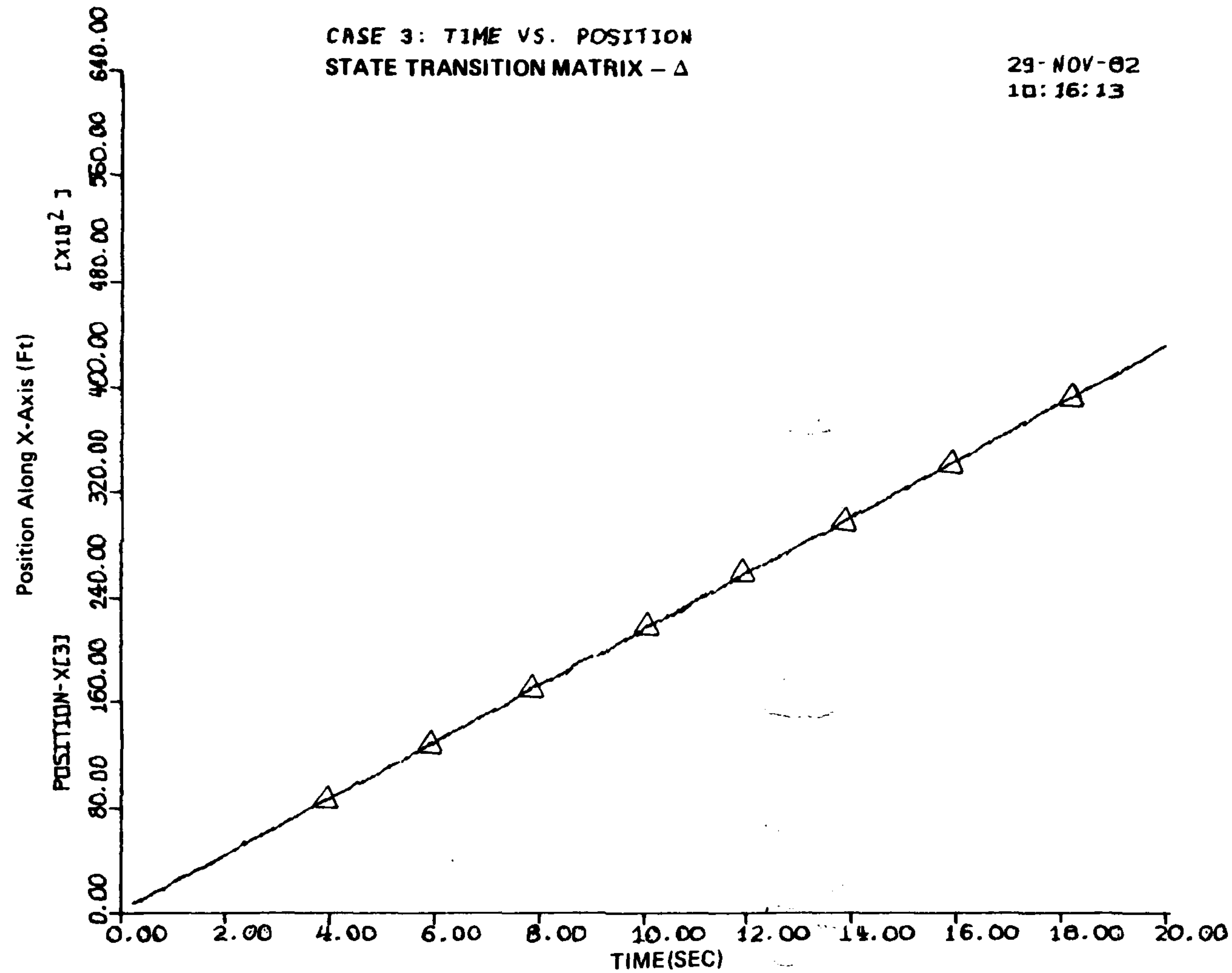

Figure 7. Target x-Position Versus Time, Case 3.

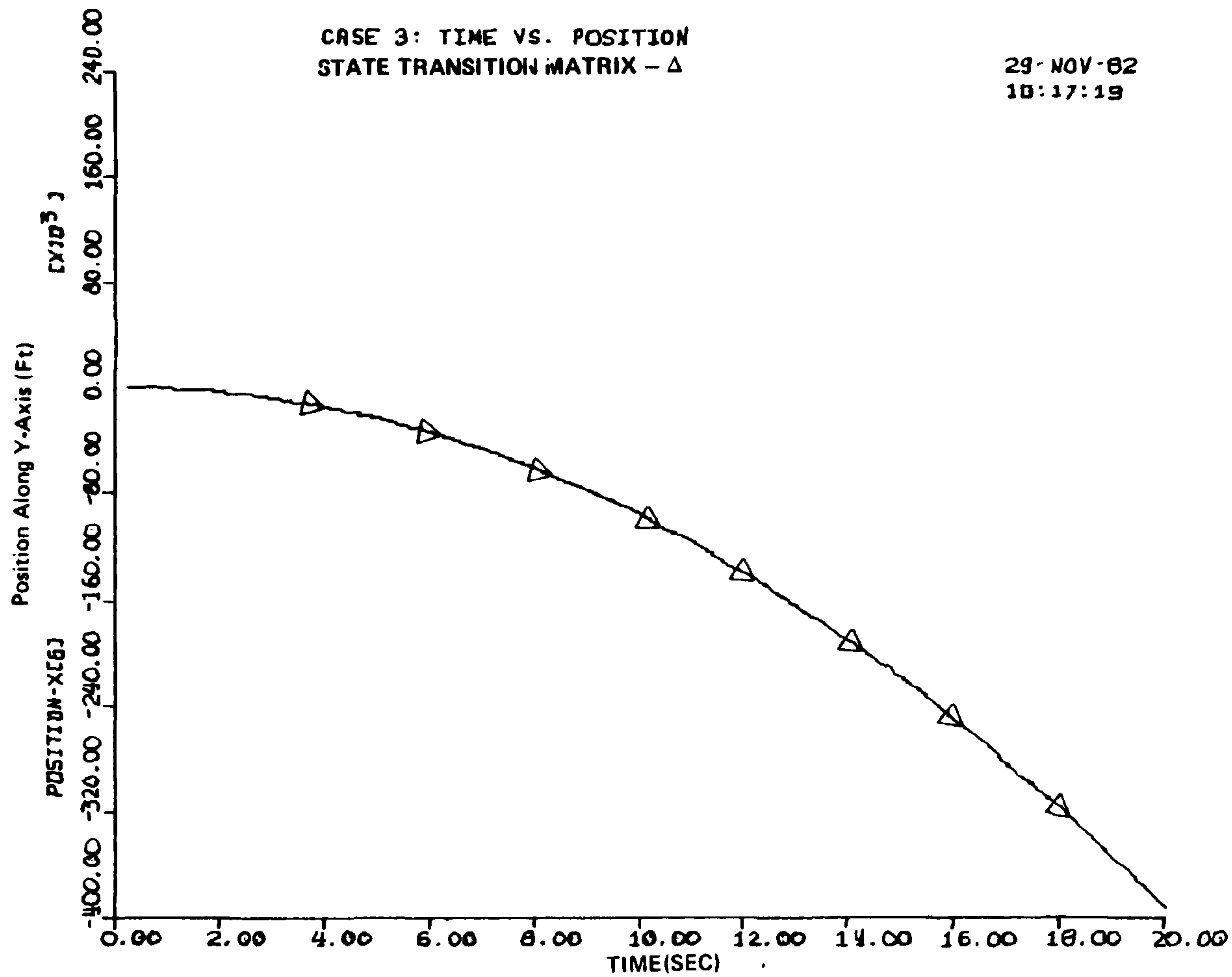

Figure 8. Target y-Position Versus Time, Case 3.

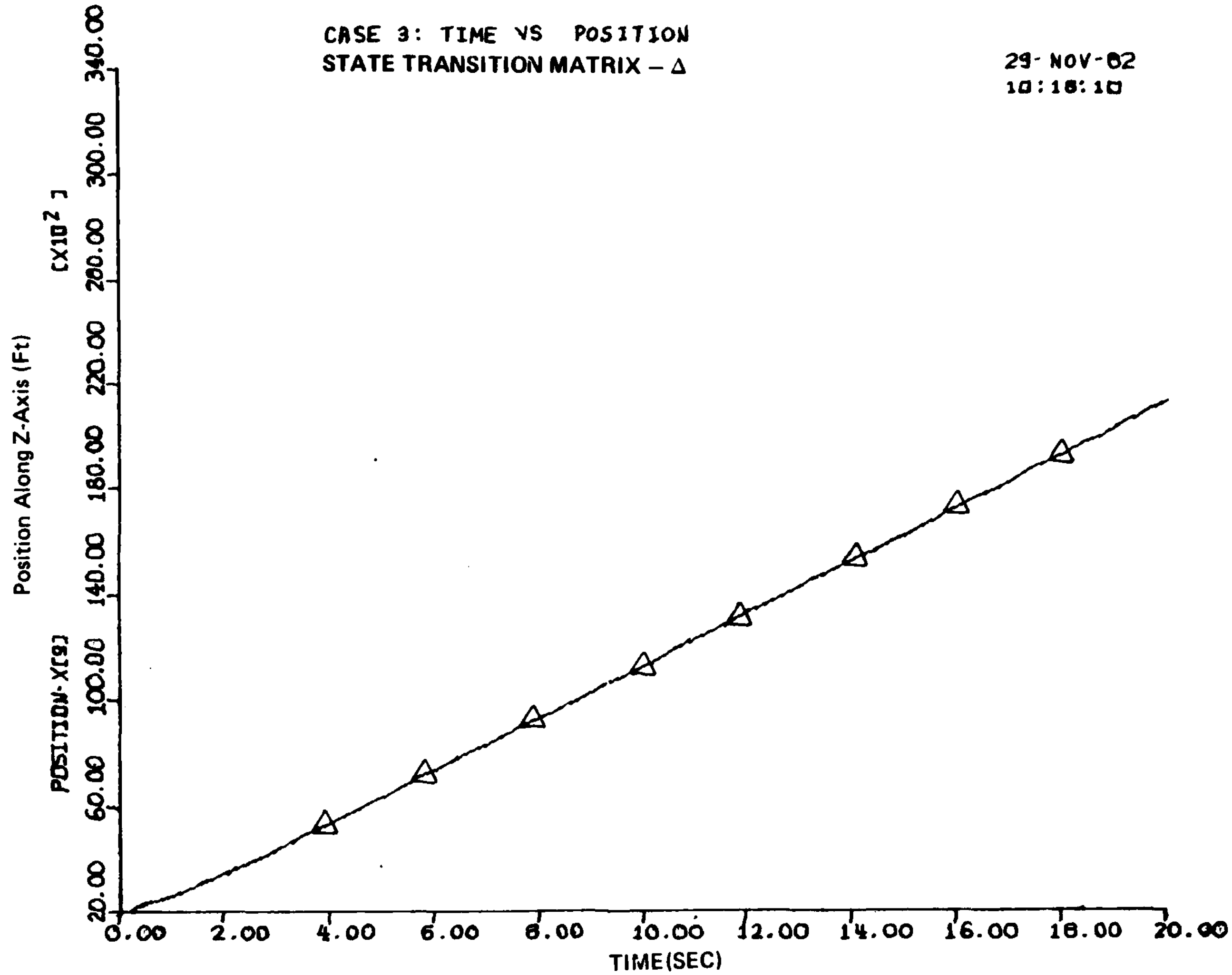

Figure 9. Target z-Position Versus Time, Case 3.

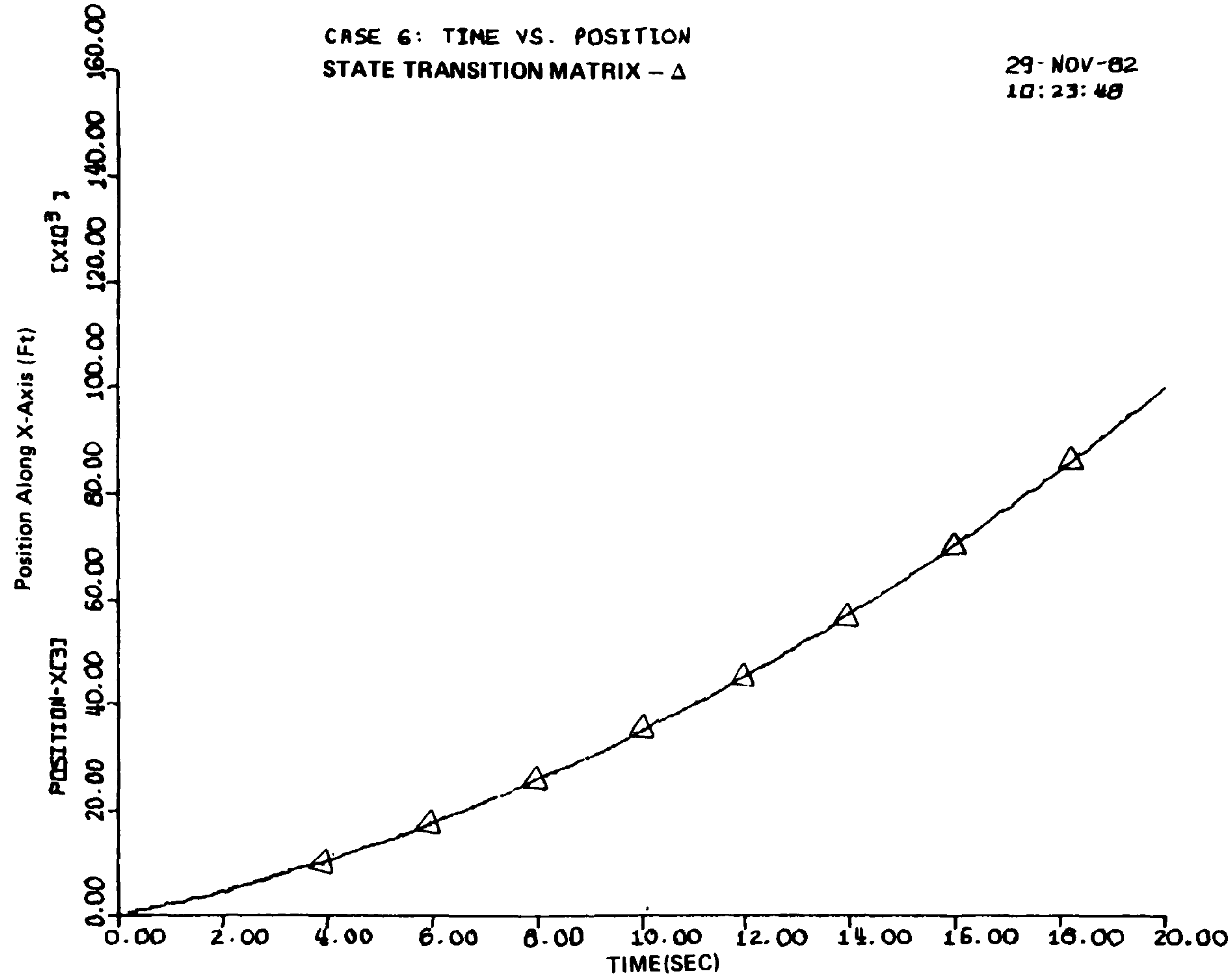

Figure 10. Target x Position Versus Time, Case 6.

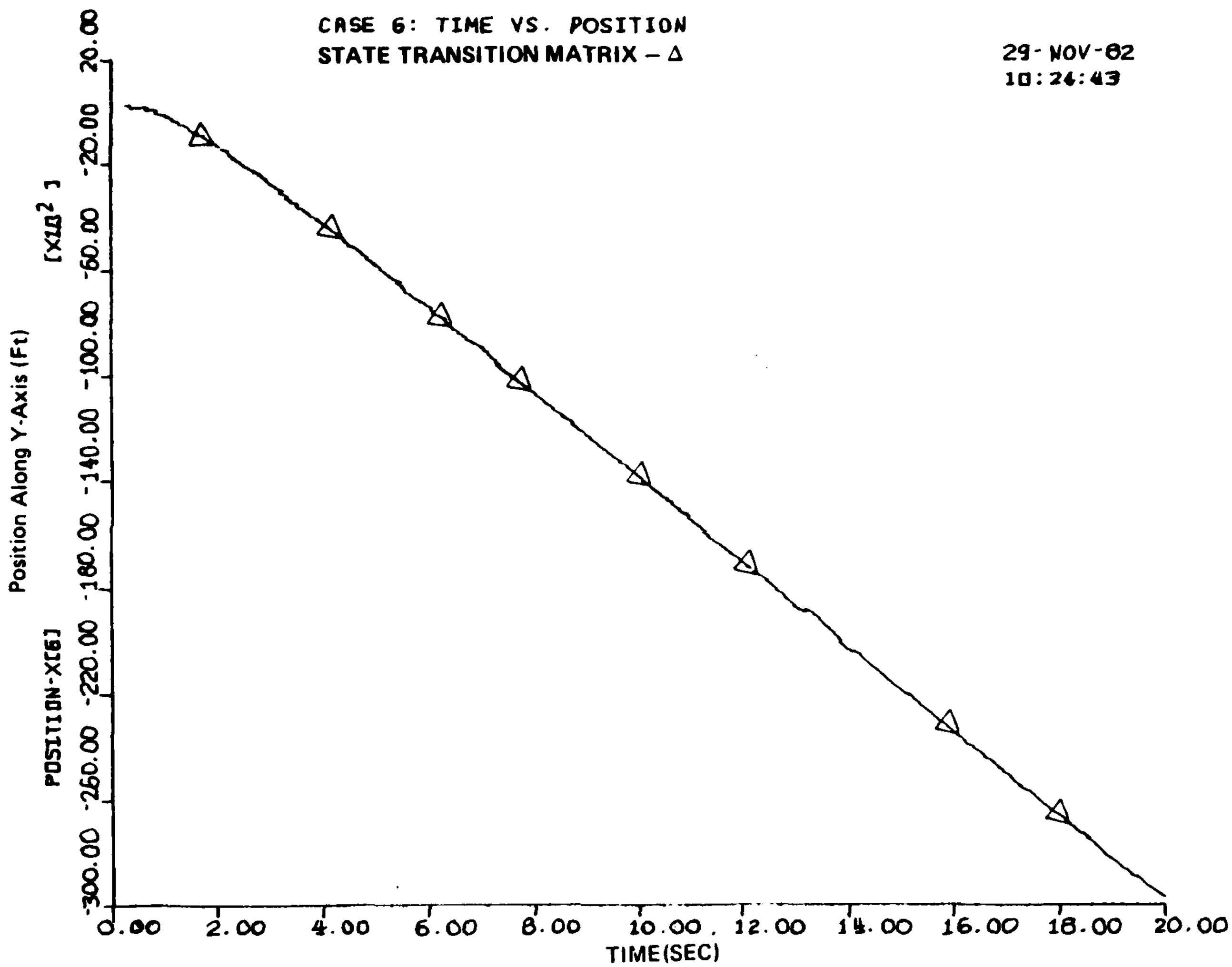

Figure 11. Target y Position Versus Time, Case 6.

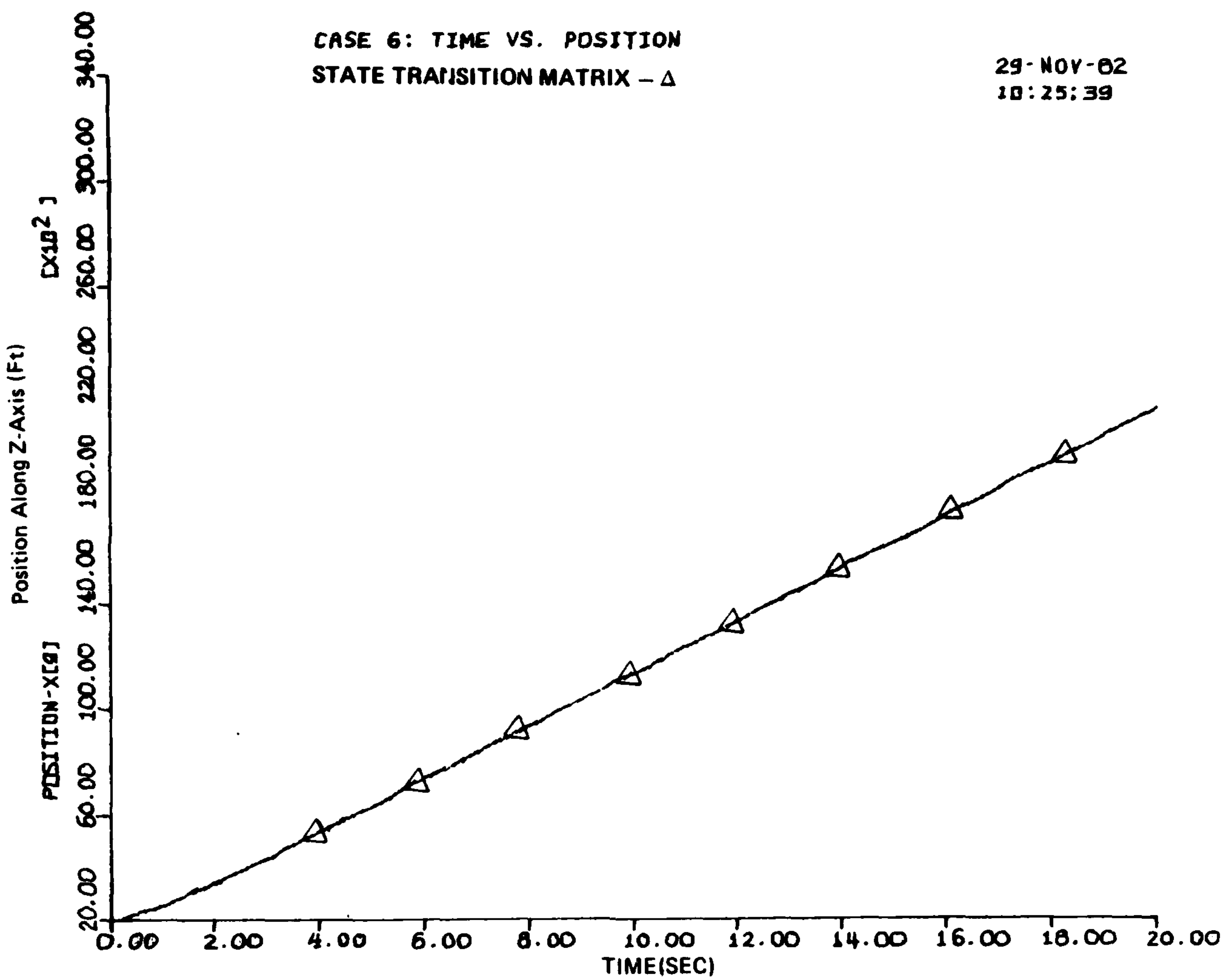

Figure 12. Target z Position Versus Time, Case 6.

APPENDIX A

DIGITAL SIMULATION PROGRAM

A listing of the digital simulation program which was used to integrate the differential equations describing the target acceleration components, Equation (2), in order to provide time histories of target position, velocity and acceleration components is presented in this Appendix. The simulation permits target state initial conditions and values for λ_x, λ_y, λ_z to be varied by the user and uses an external line-printer plot routine to provide data plot outputs.

```
C        THE FOLLOWING PROGRAM WILL ATTEMPT TO INTEGRATE THREE EQUATIONS
C        TO OBTAIN TARGET ACCLERATIONS, VELOCITIES, AND POSITIONS
C        THE PROGRAM WILL USE THE RUNGE-KUTTA 4 SUBROUTINE
         COMMON X(9),DX(9),KUTTA,DT,NX
         DATA IR,IW,IS,ID/5,6,4,1/
         CALL DATA(IR,IW,IS,ID)
         M =10
211      CONTINUE
         KOUNT =0
         KOUNT1=0
         MKOUNT = 100
         DO 102 I=1,9
         X(I)=0
102      CONTINUE
         K=0
C        READ INITIAL POSITION,VELOCITY,ACCELERATION FOR X,Y,Z
         READ(IR,*,END=241) X(3),X(2),X(1)
         READ(IR,*) X(6),X(5),X(4)
         READ(IR,*) X(9),X(8),X(7)
         TIME = 0
         NX = 9
         DT = .01
C        READ LAMBDA   XKX, XKY, XKZ
         READ(IR,*) XKX,XKY,XKZ
1000     CONTINUE
         IF (TIME.GE.20.0) GO TO 21
         K = K + 1
         DO 100 KUTTA = 1,4

         DX(1)=(-XKX)*X(1)
         DX(2)= X(1)
         DX(3) = X(2)

         DX(4)=(-XKY)*X(4)
         DX(5)=X(4)
         DX(6)=X(5)

         DX(7)=(-XKZ)*X(7)
         DX(8)=X(7)
         DX(9)=X(8)

         GO TO (30,60,30,40),KUTTA


30       CONTINUE
         IF (KUTTA.GT.1) GO TO 31
31       CONTINUE
         TIME = TIME + .5 *DT
40       CONTINUE
60       CALL RUNK
         IF (KOUNT.NE.0) GO TO 44
         WRITE (6,214)TIME,X(1),X(2),X(3)
214      FORMAT(2X,4(G12.6,2X))
         WRITE(6,216)X(4),X(5),X(6)
         WRITE(6,216)X(7),X(8),X(9)
216      FORMAT(2X,3(G12.6,2X))
44       CONTINUE
         KOUNT = KOUNT + 1
         IF (KOUNT.EQ.MKOUNT) KOUNT =0
100      CONTINUE
         IF(KOUNT1.NE.0) GO TO 1[illegible]
```

```
      WRITE (6,200)TIME,X(1),X(2),X(3)
200   FORMAT(2X,4(G12.6,2X))
      WRITE (6,205)X(4),X(5),X(6)
      WRITE(6,205)X(7),X(8),X(9)
205   FORMAT (2X,3(G12.6,2X))
199   CONTINUE
      KOUNT1=KOUNT1+1
      IF(KOUNT1.EQ.MKOUNT) KOUNT1=0
      GO TO 1000
21    CONTINUE
      GO TO 511
241   STOP
      END


      SUBROUTINE RUNK
      COMMON X(9),DX(9),KUTTA,DT,NX
      DIMENSION XA(9),DXA(9)
      GO TO (10,30,50,70), KUTTA
10    DO 20 I = 1,NX
      XA(I) = X(I)
      DXA(I) = DT* DX(I)
20    X(I) = X(I) + .5*DXA(I)
      RETURN
30    TDT = 2. * DT
      HDT = .5 * DT
      DO 40 I = 1,NX
      DXA(I) = DXA(I) + TDT * DX(I)
40    X(I) = XA(I) + HDT * DX(I)
      RETURN
50    DO 60 I = 1,NX
      VDT = DT * DX(I)
      DXA(I) = DXA(I) + 2. * VDT
60    X(I) = XA(I) + VDT
      RETURN
70    DO 80 I = 1,NX
80    X(I) = XA(I) + (DXA(I) + DT * DX(I))/6.
      RETURN
      END
```

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